‚Representing‘ in CAS-supported Mathematics Teaching

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1 Initial remarks

„*Computer Algebra Systems (CAS) are powerful tools which may help us to construct representations, navigate between different views and eventually, understand the concept more fully*.“ (Canet 1996, p. 21)

„*Using technology in teaching calculus for the past several years has done three things for me ... Most importantly, it has helped my students tie in visual and geometric conceptualisation and relate those graphics to the algebra of the calculus.*“ (Gilligan 1994, p. 20)

„*One of the main advantages of the TI-92 was a quick and easy change between different representations of mathematical objects* ...“ (Aspetsberger 1997, p. 310)

„*While some graph plotters provide a good investigate environment a real advantage of DERIVE is that algebraic manipulations can be carried out at the same time as the graph plotting...*“ (Berry/Graham/Watkins, 1994, p. 88)

This list could be expanded by numerous similar quotes.
The many forms of representation and the multitude of representations are most frequently mentioned as didactic advantage of CAS. Special terms such as „multiple linked representation“ or „window-shuttle-principle“ (cf. Heugl et al 1996) are used. There are numerous teaching examples offered, which are based on the use of different forms of representation - in particular on algebraic and graphic forms - and which concentrate on switching between these forms.

Within this paper, I will explore this „phenomenon“ more closely and deal with the question of and why it makes sense to focus in the mathematics classroom on different representation forms and various representations offered by CAS. That requires

- establishing goals and expectations which one associates with a CAS-supported mathematics classroom
- as well as an analysis of the role of representations in a CAS-supported mathematics classroom.

I will not be dealing with the first item in detail, but would like to refer to the paper of W. Peschek and E. Schneider in this proceedings (cf. Peschek/Schneider 2000). The educational philosophy of a modern general mathematics classroom which is discussed there is, in my opinion, a very suitable and secure framework for orientation.

Within this paper, I will concentrate on the second aspect and analyze the role of representations using the following sub-items:

- Didactical advantages of various forms of representation from social and individual point of view
- Representing with CAS
- Consequences of CAS-representations

2 Didactical Advantages of Various Forms of Representation from Social and Individual Point of View

I will limit myself to two aspects which are of interest to modern general teaching of mathematics:

2. 1 Expanding the Ability to Communicate

According to R. Fischer mathematics can (also) be „defined“ as:

Mathematics = representing +operating + interpreting.

The (social) efficiency of mathematics lies primarily in its function as a means of representation and interpretation, in its use as a means of expression and communication. (More detailed discussions you can find in this proceedings in Peschek/Schneider.)

Mathematics uses particularly two levels of representation for materializing (abstract) concepts, the schematic level and the symbolic level:

Schematic representations focus on some special aspects and present (abstract) relationships in such a manner that they are greatly orientated on the perceptible patterns of the context of reference, such as spatial arrangements or arrows etc. (Nonetheless one must be able to read a schematic representation in order to establish a connection to the context of reference!) What can be noted (primarily) here are tables and graphical representations. Schematic representations particularly allow a recognition of patterns and a finding of ideas, intuition and insight (cf. Fischer 1984, p. 155ff)

In symbolic representations the (abstract) relationships are expressed in formalized form (by means of symbols, signs, etc.) whose meanings are for the most part established by means of agreement. The transition from schematic to symbolic representations embodies in part an „*external expression of generalization*“ (Fischer 1984, p. 157) - a symbol stands for all the real numbers, for all points on a straight line, etc. Further, operating by rules is made possible by introducing symbols, whereby the „*degree of flexibility*“ (Fischer 1984, p. 157) is increased. The level of symbolic representations is thus that level on which the mathematical potential for solving problems lies, on which proofs are made possible, on which new knowledge is generated.

The different means of focusing make both levels of representation, the schematic as well as the symbolic, important for mathematics. But, in particular, one characteristic of mathematical working lies within the interplay between schematic und symbolic representation which R. Fischer describes as being an „*interplay between intuition, insight, gaining ideas on the one hand and syntactic justification, control, evaluation on the other hand*“ (Fischer 1984, p. 157).

Thus, in addition to the symbolic representation level (which is dominant in traditional mathematics at school) the mathematics classroom should make the schematic representation level as well as the switching from schematic to symbolic representations a main topic.
A broad offer of forms of representation and of conscious transitions among the various forms of representation also allow the efficiency of mathematics to be experienced as a means of expression and communication and provide the students with more possibilities for using mathematics in this way.

2. 2 Supporting the (Individual) Development of Meanings of Mathematical Concepts

It is not the intention of this section to enter deeper into the discussion of developing mathematical concepts, rather I would like to mention one aspect:

All mathematic knowledge needs certain representations (systems of signs or of symbols) for grasping as well as for coding the knowledge in question. These systems of signs themselves have no isolated meaning. The meaning has to be (actively) constructed by the individual him or herself. In order to „endow“ signs (symbols) with meanings, a certain (adequate) object or context of reference of a mathematical concept is necessary. The meaning of mathematical concepts is actively constructed by the individual in an interplay between sign (systems) and context of reference, in the way that possible meanings are “transferred” from a familiar or in certain aspects well-known context of reference to a new, yet meaningless system of signs. (This exposition refers to considerations of H. Steinbring who presents these relationships in the epistemological triangle - cf. figure 1 and Steinbring 1999).

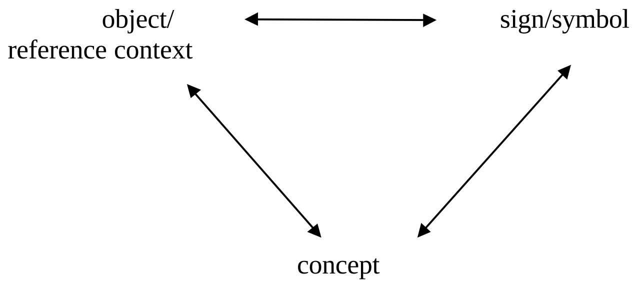

Figure 1: The epistemological triangle (Steinbring 1999, p. 42)

The use of different forms of representation and different representations is relevant for the interplay between the corners of the epistemological triangle:
Contexts of reference of a mathematical concept which are based on different forms of representation (and thus focus on specific aspects of a concept) raise the amount of what is coded or grasped by the symbol system. (One need only imagine, the symbol system $f(x)=ba^x$ on the one hand, on the other hand as contexts of reference, for example population datas with a constant percental increase over the last few years (tables), diagrams (with exponential course) of the development of the population, the average rate of increase in the population in recent years.) By an interplay the symbol system can be endowed with corresponding meanings focusing on different aspects.
Different representations (on different levels of representation) are (pre-)conditions for coding or grasping different aspects of concepts. None of the materializations present the mathematical concept as a whole; different (forms of) representation(s) focus on different aspects of a mathematical concept (and do not consider others). That means that by establishing an interplay between one reference context (object) and different systems of signs (different (forms of) representations) the „construction“ of properties of a mathematical concept can be forced (simultaneously an equalizing of the mathematical concept and the system of signs is counteracted). One imagine for example the context of reference: an annual increase in the population of a country by 8% and systems of symbols: (algebraic and recursive) function equations, tables of function values, function graphs, etc. Which of the properties of each representation can be seen to be properties of the (abstract) mathematical concept and which are constitutive for the abstract concept particularly is developed, according to W. Dörfler, in the handling with the different representations; different representations regulate themselves – socially mediated - with regard to constitutive properties (cf. Dörfler 1999).

With regard to the individual development of the (local) meanings of fundamental mathematical concepts, the mathematics classroom should force the use of different forms of representation within the field of systems of signs as well as within the area of contexts of reference.

A further aspect which is essential for the development of the meaning of concepts would be the interplay between the material (external) representations and mental (internal) constructions and the role of switching between forms of representations by the construction of mental networks of relationships. I refer therefor to papers of J. Kaput (cf. Kaput 1987, 1991).

3 Representing with CAS

The didactical advantage of different forms of representation is supported by CAS,
- by offering various forms of representation and different representations
- by the type of availability of (forms of) CAS-representation(s)

3. 1 Offering Various (Forms of) Representation(s)

CAS "know"
- on schematic level: graphical forms of representation (two and three-dimensional representations, parametric representations, polar representations, etc.) and in many cases also tabular forms of representation (above all tables of function values)
- on symbolic level: symbolic forms of representation (arithmetic and algebraic representations, recursive representations, modules) and - in limited manner - verbal forms of representation (comments, scripts, etc.)

3. 2 Availability of (Forms of) CAS-representation(s)

The offer of CAS-forms of representation does not essentially differ from the offer of „hand-drawn representations". In order to materialize concepts different forms of representation were also used without CAS. One example for this are teaching materials and schoolbooks. But, by using CAS significant facilities and simplifications are brought into the practical and flexible availability of different forms of representation. It is for the first time becoming really „efficient" to use different forms of representation:

CAS construct quite rapidly and without much effort different representations of (symbolically materialized) concepts. Proceeding on a symbolic representation, CAS construct a graph of this function "by pressing a button", "by pressing another button" they produce a table, etc. The availability of CAS-representations is closely connected with the availability of symbolic representations. In the most cases these are (pre-)conditions for producing graphical and tabular representations.

The rapid and simple construction of various forms of representation also enables and supports a rapid switching between the different forms of representation. “By pressing a button“ we can switch from a symbolic representation of a concept to a graphical one, from a table to a symbolic one, from a graphical representation to a table and so on.
The small amount of effort necessary for producing CAS-representations also have an obvious effect in carrying out changes of CAS-representations. Changes of the symbolic representation (for example changing the parameter values, the type of function, etc.) can be „transmitted“ to other forms of representation without any great amount of time and effort for the operative. Depending on the CAS the (new) constructions and the (new) calculations will be carried out automatically with the change of the symbolic representation or it is only necessary to “press a button”. The same is valid for manipulating a representation, such as changing the section of a function graph on the screen, the parameters of tables, etc.

By using CAS, representations are made accessible which could manually not be used in an adequate way because of the great amount of operative effort or of which the use were limited to the description of problems (such as recursive representations, but also graphical representations of functions in various variables and „new“ representations, such as CAS-defined or self-defined modules).

4 Consequences of CAS-representations

4. 1 Shifting of Meanings of Representations by CAS

Due to the simple and quick construction of different forms of representation the meanings of some representations have been altered. I will specify this on the basis of tables:

The meaning of tables - without CAS - lies primarily in their role as an tool for constructing function graphs. Tables also allow the easy reading of function values.
Now, CAS construct function graphs “by pressing a button”; tables are not necessary for the construction. CAS determine any function value required “by pressing a button”; it is not necessary to gather these informations from tables.
So, on the one hand, tables can be produced by CAS without any great effort “by pressing a button” and are, therefore easily accessible to the students; on the other hand they turn renouncable in their previous role!
So does handling with tables make any sense any more? And if yes, in what manner? - They do most certainly not make sense in their previous role as a (constructional) tool, but they do in the role of an “independent” form of representation as the following two examples illustrate:

- Tables have the characteristic that they can express specific quantitative patterns more clearly and more immediately than do other forms of representation. Using a table, one can for example „observe“ a constant absolute or a constant relative increase by comparing the values in the table in an elementary way. (A corresponding interpretation of the representation of the term requires knowledge of interpretation of the structure of the algebraic expression; the function graph primarily targets qualitative patterns.)
- Tables can also contribute to (operative) problem solving. In such cases, where the performance capacity of CAS has reached its limits or when these limits have been exceeded one can nevertheless be successful in solving the problem by means of such elementary methods as tables. As an example of this let me mention the solving of complicated equations or inequations, in particular, when recursive descriptions of concepts are given. By using functional concepts and producing corresponding tables one can reach an (approximative) solution in such cases in a short period of time.

4. 2 Shifting of Required Knowledge and Skills

On the one hand, producing representations by CAS reduces the students’ operative activities (and the corresponding required knowledge and skills), on the other hand, it requires (traditional as well as extended) competences from the students in other fields:

- Extended competences in the field of the <u>symbolic:</u>
  * in part, new conventions with regard to the use and meaning of mathematical signs and of CAS-specific (series of) symbols due to syntax being determined by CAS
  * recognition of the existing term structure and knowledge about the hierarchy of calculation operations for the correct input of the term as well as for recognizing the equivalence of that which is shown on the screen to that which is shown on „paper“, for example

$$B = \frac{R}{(1+i)^n} \cdot \frac{(1+i)^n - 1}{i} \qquad \leftrightarrows$$

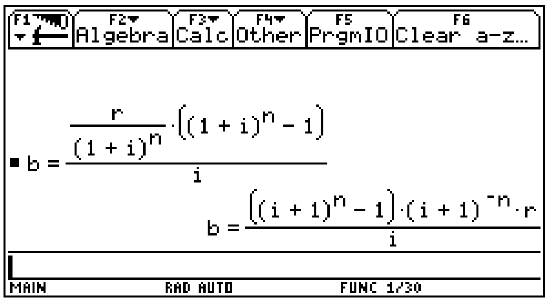

Figure 2

- Greater emphasis of a <u>functional point of view</u>. In particular
  * the graphical or tabular representation bases on a description (and input) in form of function equations.

* the possibility offered by CAS to store terms as modules is based on emphasizing the functional point of view of mathematic formulas (such as *cone(r,h)* - volume formula for cones*; ew(r,i,n)* - the final value formula of annuity)

- Interpreting
  * The representations produced by CAS are first meaningless signs which need to be interpreted in context. The interpretation of the representation is carried out
    - on an inner mathematical level (solution of an equation; derivative of a function, integral, graph of a function, etc.)
    - in context (optimal production program, rate of change, area underneath a curve, course of a cost function, etc.)
    - evaluation of the representation: control (for example: "Can such a high/low value be correct?") and reflection (for example: "How many solutions can this equation have maximum, and how many in the given context?")
  * "Translations" between the different forms of representation
  * But: no interpretation work is necessary in connection with operating by rules (for example interpreting of the term structure in order to decide which derivation rule is to be used). This all will be taken over by CAS.

- (Traditional) mathematical basic knowledge is required, in particular
  * for the input of symbolic expressions (see above)
  * for the reflection and interpretation of CAS-representations in context, such as
    - basic knowledge of specific properties of types of equations, in order to be able to evaluate solutions in context
    - basic knowledge about types of functions and term structure, in order to be able to differentiate between the local and global course of the functions as well as between the properties of the function and the properties „produced" by the chosen scale when considering the function graphs shown on the screen.

5 Closing remarks

CAS supports the use of different forms of representation in the mathematics classroom. This should be used in an adequate didactical way for the development of basic skills in the field of representation and interpreting as well as for the development of the basic understanding of mathematical concepts and for reflections. The goal of a modern generally educating mathematics teaching cannot be - as has been observed in numerous curricula and teaching projects - to use the representational possibilities of CAS primarily for visualizing operative procedures and methods which would (and should) in any case be outsourced to CAS.

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